4-Dimensional Graphing of Complex Numbers

by

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ABSTRACT

A computer program was written to graph the real and imaginary solutions to various equations in four dimensions in the form y+zi=f(x+ti). A number of interesting relationships involving imaginary numbers were determined from these equations, including the demonstration that a hyperbola is actually an imaginary ellipse. Several proofs were developed to explain the relationships, as well as a new method of graphing three-dimensional equations (such as a sphere) using imaginary numbers. A relationship for fourth-degree equations similar to the x2=(-x)2 relationship for second-degree equations was also derived.

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SUMMARY

A computer program was written to graph the real and imaginary solutions to various equations in four dimensions. A number of interesting relationships involving imaginary numbers were determined from these equations, including the demonstration that a hyperbola is actually an imaginary ellipse. Several proofs were developed to explain the relationships, as well as a new method of graphing three-dimensional equations (such as a sphere) using imaginary numbers.

A NOTE ON NOTATION

Throughout this paper, there will be some strange notation used, mostly due to wordprocessing difficulties. The following is a summary of such notation:

Symbol	Denotes
SQR(x)	Square root of x
Nth(n)	Nth root of n
y/x+z	y divided by (x+z)
(y/x)+z	(y divided by x)+z
cis(a)	cos(a) + isin(a)

INTRODUCTION

Since their invention, complex numbers (number which contain SQR(-1)) have gotten little attention in terms of graphing. Many people know how to solve equations with complex numbers; however, there have been no ways to graphically show the relationships involved within these formulas, while remaining true to the common Cartesian system of graphing.

The method used herein for graphing the complex numbers is as follows. Many equations can be put into the form y=f(x), where f(x) is some expression. However, there are a number of f(x) which, for certain values, will return complex numbers. So, we can use the form y+zi=f(x), in order to account for such numbers. One might then wonder what complex values for x would do, and so we use the form y+zi=f(x+ti), where t is time. This will create animated graphs of common equations.

THE PROGRAM

Now that we have our graphing theory, we need a way to graph. The most obvious solution to this problem would be the computer- in this case, a Compaq Deskpro 286e. Since there does not seem to be an abundance of 4-D calculating and graphing programs on the market, it would seem that prudent thing to do is to create one. In this case, QBasic was the language of choice, primarily because it comes with the OS. To avoid lengthy parsing operations, the program was written devoid of any significant user interface; in order to change it, you had to change the program. Things such as scaling, viewmode, and detail could be changed from within the program; however, in order to change the equation, the program must be edited.

The program works in the following manner. The equation is located in a subroutine (named FINDY), which could be easily edited. It then substitutes sequential values of X and T, incrementing the values with the corresponding variable D (which can be changed easily while the program is running) up to the preset ranges. For example, if DX=.5, DT=.5, the X range is 2, and the T range is 1, the following values would be substituted into the equation:

0 + 0i	0.5 + 0i	1.0 + 0i	1.5 + 0i	2.0 + 0i
0 + .5i	0.5 + 0.5i	1.0 + 0.5i	1.5 + 0.5i	2.0 + 0.5i
0 + 1.0i	0.5 + 1.0i	1.0 + 1.0i	1.5 + 1.0i	2.0 + 1.0i

In order to get corresponding Y and Z values, the computer had to be able to handle imaginary numbers. To do this, subroutines were created to handle the mathematical operations needed.

Addition

Addition of imaginary numbers is very simple, and does not even need a subroutine to handle the addition. Due to the fact that the imaginary parts and real parts of complex numbers do not affect each other when added, straight addition was used.

Multiplication

The multiplication algorithms were derived in the following way, where the numbers being multiplied are x+yi and m+ni.

$$(x+yi)\,(m+ni)$$

$$xm + ymi + xni + ynii$$

$$xm + ymi + xni + yn(-1)$$

$$(xm - yn) + i(ym + xn)$$

Using this formula, a subroutine was easily written to take two imaginary numbers and multiply them.

Division

The division algorithms were derived much in the same way that the multiplication algorithms were.

$$\frac{(x+yi)}{(m+ni)}=\frac{(x+yi)(m-ni)}{(m+ni)(m-ni)}=\frac{xm+ymi-xni+yn}{mm+mni-mni+nn}=\frac{(xm+ym)+(ym-xn)i}{m^2+n^2}$$

Exponents

The easiest way to go about raising imaginary numbers to a given power is Demoivre's theorem, and that is the method which was used here. Rather than present an explanation of the subroutine here, a complete listing of it, along with comments, can be found in Appendix A.

A number of questions may be raised about this subroutine, and so some answers are in order. First, the reason for rounding off the answers is simple, the computer is not 100% accurate, and so will return values off by a small amount. Second, the shortcuts at the beginning speed up processing time in some equations.

Demoivre's Theorem

Demoivre's Theorem states that the value of $(x+yi)^n$ can easily be computed by converting the complex number into polar form (rcis th). Once the polar form of the number is gotten, all one needs to do is raise the coefficient r to the power n, and multiply the angle th plus 360 times any positive whole number by the power n. If the number n is a fraction, then there will be several roots (for example, 1/3 of 90 is 30, but 1/3 of 450 (360 + 90) is 150, which would be a second root. 1/3 of 810 (720 + 90) would be the third

The IMEXP routine mentioned earlier differs in two respects. First, it replaces the COS(th) with SIN(th), in accordance to Demoivre's Theorem. Second, the shortcuts in the beginning were changed slightly for cases involving imaginary numbers (see the program listing for more information).

Absolute Value

The absolute value function is nearly as simple as the addition function. Because the absolute value of a number is the distance from the origin, all one must do is use the Pythagorean Theorem to find out that:

$$ABS(y+zi)=SQR(y^2+z^2)$$

Plotting

The program offers the following viewmodes: DATA, XY, YZ, XZ, 3D, and Composite. The data mode displays the X, Y, and Z values in a list. The XY, YZ, and XZ modes show corresponding views (the XY plane is the one we are accustomed to). The 3D view shows a 3-dimensional representation, where the Y axis is vertical, the Z axis is horizontal, and the X axis comes out at a 45 angle (using the coordinates (z+x/sin30,y+x/sin30). The reason that this view was chosen rather than one with the Z axis going upwards is that this image looks more like the XY image, which is the standard for 2-dimensional Cartesian graphing. The composite view shows the four graphical views, each taking up one fourth of the screen (this is the viewmode from which the data was taken). For more information on the composite viewmode, see appendix A.

RESULTS

The line

The line, which comes in the form y=mx+b, is not noticeably different when graphed in 4D. Because there is no conceivable way to get imaginary output from this equation without imaginary input, the t=0i frame of the graph has no imaginary solutions. As time goes on, the line moves away from the screen (because the real and imaginary parts of the data do not affect each other, the line moves one Z unit for one T unit).

The ellipse and hyperbola

For simplicity of formula, rather than graphing a more complicated ellipse, a circle was graphed (an ellipse with eccentricity=1). The formula used was y=SQR(625-x^2). No scaling was enacted. The graph offered, at first glance, some very curious results. The circle had an imaginary curve on each of its x-intercepts (see figure 1).

What is happening may not be immediately obvious. However, after graphing a hyperbola, it was interesting to see the imaginary circle connecting the parts of the hyperbola, at the x-intercepts (see figure 2). The reason for this is rather simple, and can be shown through simple equation-manipulation.

ELLIPSE	HYPERBOLA
$x^2/a^2 + y^2/b^2 = 1$	$x^2/a^2 - y^2/b^2 = 1$
$y^2/b^2 = 1-x^2/a^2$	$y^2/b^2 = x^2/a^2 -1$
$y/b = SQR(1-(x^2/a^2))$	$y/b = SQR((x^2/a^2)-1)$
$y = b\ SQR(1-(x^2/a^2))$	$y = b\ SQR((x^2/a^2)-1)$

Now, what the evidence insinuates is that an ellipse is an imaginary hyperbola, and vice versa. So if we multiply each equation by i, like so:

$y= bi\ SQR(1-(x^2/a^2))$	$y = bi\ SQR((x^2/a^2)-1)$
$y= b\ SQR(1-(x^2/a^2))\ SQR(-1)$	$y= b\ SQR((x^2/a^2)-1)\ SQR(-1)$
$y= b\ SQR(\ (-1)\ (1-(x^2/a^2))$	$y= b\ SQR(\ (-1)\ (x^2/a^2)-1)$
$y= b\ SQR((x^2/a^2)-1)$	$y= b\ SQR(1-(x^2/a^2))$

we find that the equations have switched places.

A modification of the circle

One might wonder what would happen if we took our circle, $y=SQR(625-x^2)$, and changed the powers of 2 into various other powers (for example, $y=6th(625-x^6)$). This results in what one might call a 'squashed' circle (see figure 3). A more interesting equation, however, would be $y=2Nth((625-x^2)^N)$, for which the graph consists of n circles, at angles of 360/n (see figure 4). This situation is explained in more detail below (see the Theorem of Imaginary Duplication).

A conceptional problem

There seems to be something of a conceptional difficulty when having only the imaginary outputs displayed at a given time. For example, when one rotates a hyperbola, one would expect to still have a rotated ellipse in the middle. However, take the 45 rotation of a hyperbola, $y=1/x$. As long as one is limited to real inputs ($t=0$), there will be no imaginary output. It would seem that our imaginary ellipse has disappeared. However, take the hyperbola $x=SQR(y^2-625)$. This hyperbola also will have no imaginary ellipse if one graphs by substituting x values and deriving y values. However, if y values are substituted, imaginary x values will be received (meaning t<>0). This implies that the ellipse exists has been shifted through various time values. This can also be applied to the y=1/x equation.

THE "SIMPLE ROOT CURVES"

The "simple root curve" (SRC) is defined as any curve which can be expressed in the form y=nth(x). After doing graphs of many different SRCs, it was determined that they could be summed up into three categories, depending on the value of 1/n (whic from here on will be referred to as a). These three categories are: a is an odd number, a is divisible by four, and a is an even number not divisible by four.

NOTE: In the following discussions, any reference to a number n refers to the denominator of the exponent of the SRC.

ALSO NOTE: For proofs of the relationships discussed below, see Appendix B.

A IS AN ODD NUMBER

The thing one will notice when looking at an odd-numbered SRC is that, when viewed from he XY plane, quadrants II and III are a 180° rotation of quadrant I and IV (see figure 5). This implies a relationship which was already known; that if a number X has a root a+bi, the number -X will have a root -(a+bi).

One of the nice things about SRCs is that they are easily graphed once the real portion of the curve is obtained. This portion can be obtained in many ways, ranging from simple substitution and graphing, to Calculus. The odd-numbered curves consist of n rotations of the real portion around the x-axis, set 360°/n apart.

A IS AN EVEN NUMBER NOT DIVISIBLE BY FOUR

When viewing one of these SRCs, one will notice two things: that the graph is symmetrical over the X-axis, and that the XZ view is a 180 rotation of the XY view (see figure 6). This implies two relationships: That if a number X has a root a+bi, it will also have a root -(a+bi), and that the number -X will have roots b+ai and -(b+ai).

The method for graphing all even-numbered SRCs (included those discussed below) is as follows. These graphs are divided left ($x<0$) and a right ($x>0$) sections. The right section consists of n/2 rotations of the curve around the x-axis, every 720°/n. The left section consists of the right section, reflected over the y-axis, and rotated around the x-axis 360°/n.

A IS DIVISIBLE BY FOUR

This is by far the most interesting curve. By looking at the graphs, the first thing one would notice is that the XZ view and XY view seem to be the same. Not only that, but they seem to be symmetrical with respect to the x-axis (see figure 7). This implies the following: That if for a number X, if one root is a+bi, three other roots will be -(a+bi), b+ai, and -(b+ai). The method for graphing of these SRCs is exactly the same as that of graphing other even-numbered SRCs (see above).

The Theorem of Imaginary Duplication

When examining the graphs of various equations, one will notice the following relationship. Given the equation y=f(x), if we then take the equation $y=\text{Nth}(f(x)^N)$, we will notice it graphs in the following manner: There will be N rotations about the x-axis of the original equation, set apart at angles of 360°/N. This would also imply interesting things about the graph of $y=\lim_{N\text{->infinity}} \text{Nth}(f(x)^N)$, there would be an infinite number of rotations of the graph, infinitely close together, hence a 'solid object'. (NOTE: For simplicity, the expression $\lim_{N\text{->infinity}} \text{Nth}(f(x)^N$ will be replaced with the function IDP(f(x))).For example, the graph of y=IDP(x) would be a 3-dimensional circular cone, the graph of $y=\text{IDP}(x^2)$ would be a 3-dimensional elliptic paraboloid, and the graph of $y=\text{IDP}(b\text{SQR}((x^2/a^2)-1))$ would be an ellipsoid connecting a hyperboloid of two sheets.

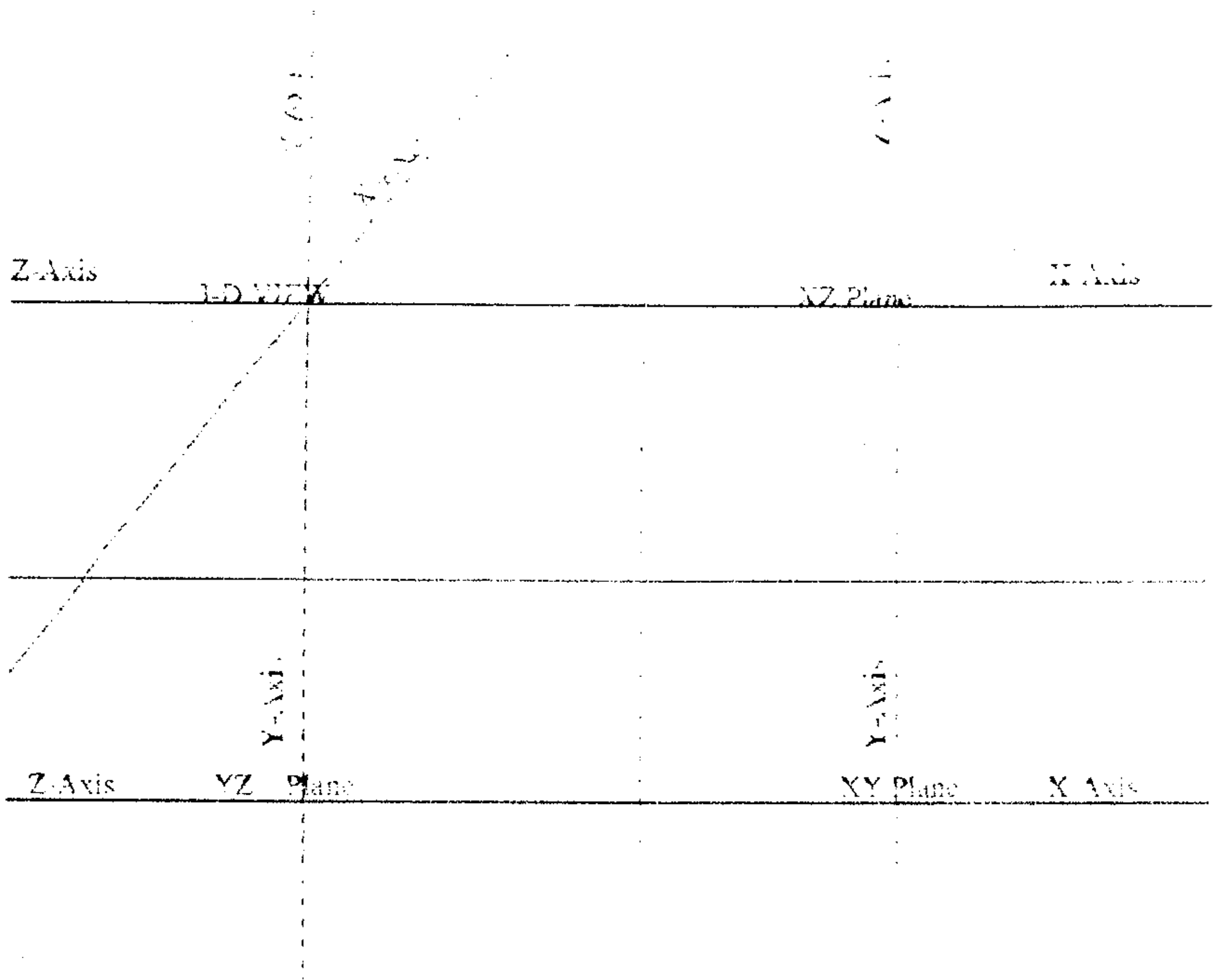
The proof of this goes as follows. Demoivre's theorem (see explanation above) states that when the roots of a polar number are taken, the angle of the number in polar form, th, is multiplied by the exponent, r, and then added to 360nr, where n is any positive whole number. The significant part of this is the 360nr, which will return angles which differ by a factor of 360r. Since r in the case if IDPs is the reciprocal of the exponent N, we have angles separated by a factor of 360°/N.

CONCLUSIONS AND OPPORTUNITIES FOR FURTHER RESEARCH

A number of interesting relationships could be determined from the graphing of complex numbers. However, one should note that the information discussed in this paper is only the tip of the iceberg. More complicated equations such as $x^3+x^2+x \quad =y^3+y^2+y+1$ could be graphed (perhaps using a formula for solving cubic equations, then substituting various values for x). A system for handling complex numbers in trigonometric functions (such as SIN and TAN) could be applied. There was also no discussion of complex numbers as exponents, which would also make for interesting graphs. Lastly, the fourth dimension of theses graphs was not explored, other than to see if the theorems for the SRCs held true for complex inputs.

APPENDIX A: Graphs and Listings

[illegible]



COLOR CODING

The color coding works in the following fashion, depending on the signs of X, Y, and Z (in other words, depending on the octant of the point.

|X|Y|Z|COLOR
[illegible]
+|+|-|RED
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]

FIGURE 1: y=SQR(625-x^2)

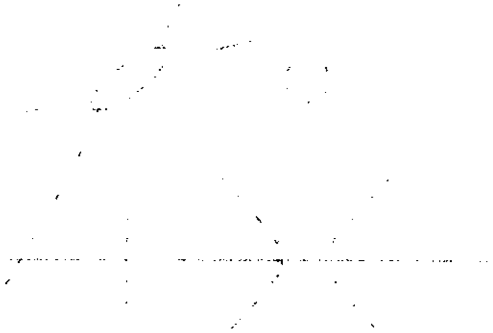


FIGURE 2: y=SQR(x^2-625)

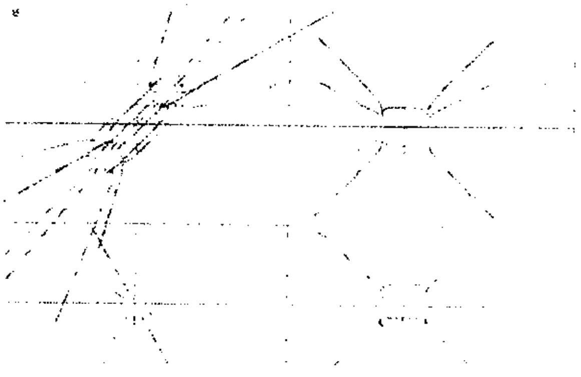


FIGURE 3: y=[illegible](25^2-x^2)

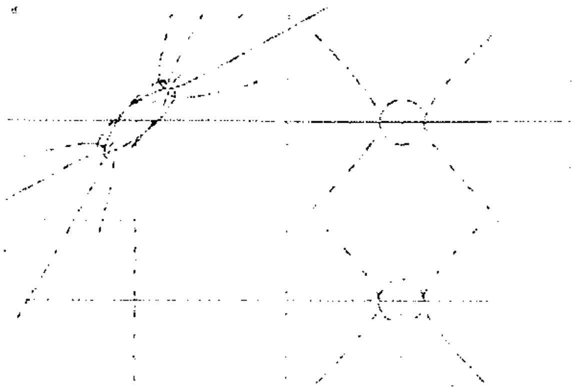


FIGURE 4: y=SQR((SQR(625-x^2[illegible]

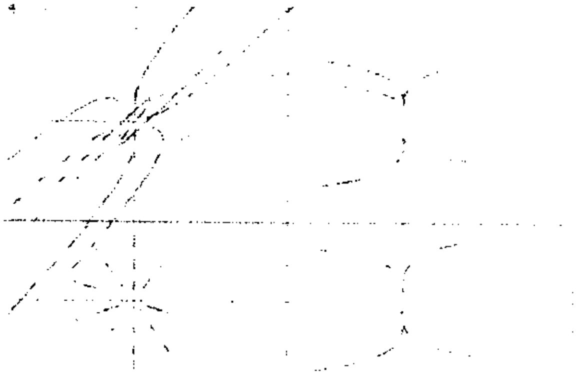


FIGURE 5: y=5th(x)

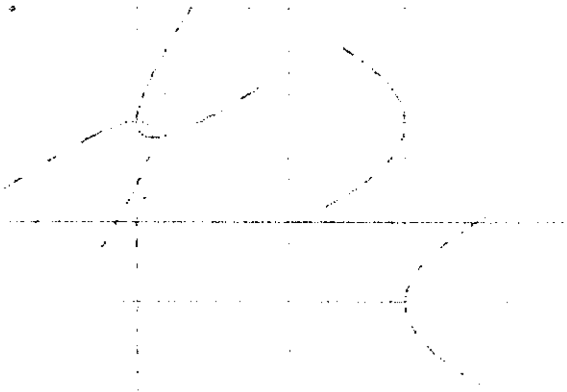


FIGURE 6: y=SQR([illegible]

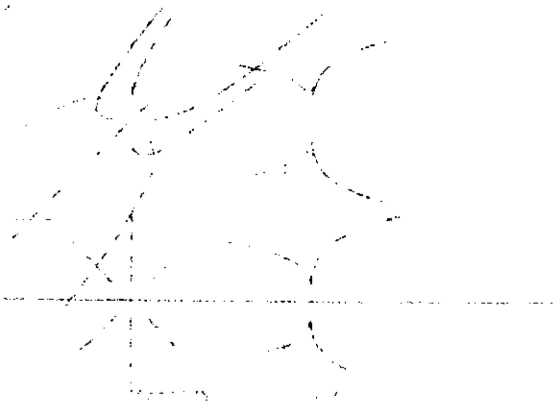


FIGURE 7: y=4th(x)

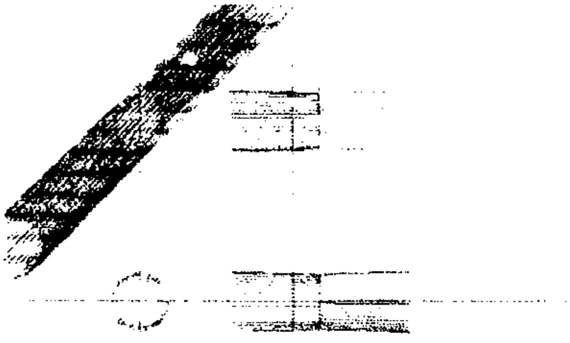


FIGURE 8: y=42nd(x)

```
FUNCTION rmexp (x, y, a, c)                         'Input in the form
                                                    '$(x+yi)^a$, where
                                                    'c is the desired answer
                                                    '(see below)
IF x = 0 AND y = 0 THEN rmexp = 0: GOTO bend        'If both x and y are zero,
                                                    '$(x+yi)^a$ must also
                                                    'equal zero.
IF x = 0 THEN x = .0000001                          'Because the tangent ratio
                                                    'involves division by x, we
                                                    'must set x to a value very
                                                    'close to x, to avoid
                                                    'division by 0.
IF a = 0 THEN rmemp = 1: GOTO bend                  'Any number to the 0
                                                    'power results in 1.
IF a = 1 THEN rmexp = x: GOTO bend                  'Any number to the 1
                                                    'power is itself.
m = SQR(x ^ 2 + y ^ 2)                              'Use the distance formula
                                                    'to get the distance.
th = ATN(ABS(y / x))                                'Find the reference angle.
IF x <= 0 AND y > 0 THEN                            }
        th = tpi / 2 - th                           } Statement to find the
ELSEIF x < 0 AND y <= 0 THEN                        } quadrant of the angle,
        th = tpi / 2 + th                           } and the add the
ELSEIF x >= 0 AND y < 0 THEN                        }appropriate value.
        th = tpi - th                               }
END IF                                              }TPI=2(pi)

m = m ^ a                                           } Statements to
th = th * a + ((tpi * c * a))                       } follow through with
rm = m * COS(th)                                    } Demoivre's theorem.

IF rm < 0 THEN neg = 1: rm = ABS(rm)                }
rm = INT(rm * acc) / acc                            } Round off the answer.
IF neg = 1 THEN neg = 0: rm = 0 - rm                }
rmexp = rm                                          'Return the value.
bend:
END FUNCTION
```

APPENDIX B: PROOFS

PROOF 1: $(a+bi)^{2n}=(-a-bi)^{2n}$

This realationship can easily be shown by proving $(a+bi)^2=(-a-bi)^2$, like so:

$$(-a-bi)^2$$

$$(-1)^2\,(a+bi)^2$$

$$(1)\,(a+bi)^2 = (a+bi)^2$$

Now that this relationship has been determined, it should be obvious that raising the entire equation to the n power will not affect its accuracy.

PROOF 2: $(a+bi)^{4n}=(-a-bi)^{4n}=(b-ai)^{4n}=(-b+ai)^{4n}$

In order to show this relationship, first it will be proven for:

$$(a+bi)^4=(-a-bi)^4=(b-ai)^4=(-b+ai)^4$$

This proof can be divided into two parts: The first parts involves proving the first two expressions equal, and the last two expressions equal. Because the equation $(a+bi)^4=(-a-bi)^4$ can be expressed as $[(a+bi)^2=(a-bi)^2]^2$, this half of the proof has already been proven above (proof #1). Note that the same logic can be applied to $(b-ai)^4=(-b+ai)^4$.

The second half is a bit more complicated, but can easily be shown using binomial expansion.

$$(a+bi)^4$$

$$a^4 + 4a^3\,bi + 6a^2\,(bi)^2 + 4a\,(bi)^3 + (bi)^4$$

$$a^4+ 4a^3\,bi + 6a^2\,b^2\,i^2 + 4a\,b^{\wedge 3}\,i^3 + b^4i^4$$

$$a^4+ 4a^3\,bi + 6a^2\,b^2\,(-1) + 4a\,b^3\,(-1)i + b^4\,(1)$$

$$a^4+ 4a^3\,bi - 6a^2\,b^2 - 4a\,b^3\,i + b^4$$

$$(b-ai)^4$$

$$b^4- 4b^3\,ai + 6b^2\,(ai)^2 - 4b\,(ai)^3 + (ai)^4$$

$$b^4- 4b^3\,ai + 6b^2\,a^2\,i^2 - 4b\,a^{\wedge 3}\,i^3 + a^4i^4$$

$$b^4- 4b^3\,ai + 6b^2\,a^2\,(-1) - 4b\,a^3\,(-1)i + a^4\,(1)$$

$$b^4- 4b^3\,ai - 6b^2\,a^2 + 4b\,a^3\,i + a^4$$

$$a^4+ 4a^3\,bi - 6a^2\,b^2 - 4a\,b^3\,ai + b^4$$

When expanded, $(a+bi)^4=(b-ai)^4$, and through the chain rule, $(-a-bi)^4=(a+bi)^4=(b-ai)^4=(-b+ai)^4$. It is also interesting to note that this can be proven using Demoivre's theorem, and physically shown by graphing the complex numbers on a 2-dimensional plane.

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